Adopting the usual definition of division as the inverse of multiplication, we have readily:

V. Division: $$\frac{ar+bi+cs}{\alpha r+\beta i+\gamma s}=\frac{a\alpha+b\beta}{\alpha^2+\beta^2}r+\frac{-a\beta+\alpha b}{\alpha^2+\beta^2}i+\frac{c}{\gamma}s.$$

That the formal laws of ordinary algebra hold for multiplication and division is evident from the form of the results in IV and V.

So far we have not brought our new numbers into relation with ordinary real numbers. The definitions of addition and multiplication enable us to do this. By the formula for addition we have

$$m(ar+bi+cs)=mar+mbi+mcs.$$

The multiplication formula gives the same result for the product,

$$(ar+bi+cs)(mr+0i+ms)=mar+mbi+mcs.$$

Hence we shall define

VI. $$m=mr+0i+ms,$$

and all real numbers are to be combined with triple numbers in accordance with this definition.

We have now to develop the zero-product theorem. Set

$$(ar+bi+cs)(\alpha r+\beta i+\gamma s)=(a\alpha-b\beta)r+(a\beta+b\alpha)i+c\gamma s=0.$$

This requires that

$$a\alpha-b\beta=0,\quad a\beta+b\alpha=0,\quad c\gamma=0.$$

There is no difficulty in verifying that these relations can be satisfied only when one of the factors is of the form $0r+0i+0s$ or when one of them is of the form $0r+0i+cs$ and the other is of the form $\alpha r+\beta i+0s$. We have then the following theorem:

THEOREM. *If the product of two factors is zero, then one of them is zero; or, one of them is of the form* $0r+0i+cs$ *while the other is of the form* $\alpha r+\beta i+0s$.

There are thus in this system three quantities playing in some respects the rôle of a zero.

It is now easy to see that division is unique and determinate *except in three cases,** namely,

VII. $$\frac{0r+0i+0s}{0r+0i+0s'} \quad \frac{0r+0i+cs}{0r+0i+\gamma s'} \quad \frac{ar+bi+0s}{\alpha r+\beta i+0s}.$$

The first is the indeterminate 0/0 of our ordinary complex algebra. The other two have an easy geometric interpretation. For the second, the distance of the point from the plane is fixed, but its projection on the plane is indeterminate. For the third, the projection on the plane of reference is fixed, but the distance from the plane is indeterminate.

If one wishes to interpret for ordinary complex algebra any results obtained by the use of this triple algebra he has merely to omit the terms containing s and to set $r=1$. The formulae and results then become those of ordinary complex algebra.

Our new algebra is now completely defined and related to our former algebra. It is now apparent that *the only added complication or difficulty which can possibly arise will be due to the existence of three indeterminates in division instead of one as in the former algebra.* That this is by no means an insurmountable difficulty becomes apparent as our work progresses.

§2. Functional Dependence.

We shall employ the notation

$$R \equiv xr+yi+zs,$$
$$S \equiv ur+vi+ws,$$

and shall say that S is a function of R when S may be obtained by performing any set of operations upon R considered as a single whole. This of course is precisely analogous to the ordinary conception of a function of a complex variable. Moreover we shall further assume that the functions with which we deal have a differential coefficient.

Then if $S=f(R)$, where the form of f is not specified, we have

$$ur+vi+ws=S=f(R)=f(xr+yi+zs).$$

Differentiating with respect to x, y, z, we obtain

(1) $$\frac{\partial S}{\partial x}=\frac{dS}{dR}\frac{\partial R}{\partial x}=rf'(R)=r\frac{dS}{dR},$$

*Of course this presupposes the introduction of infinity. This is not expressly defined here because it introduces nothing new.

$$\frac{\partial S}{\partial y} = \frac{dS}{dR}\frac{\partial R}{\partial y} = if'(R) = i\frac{dS}{dR}, \tag{2}$$

$$\frac{\partial S}{\partial z} = \frac{dS}{dR}\frac{\partial R}{\partial z} = sf'(R) = s\frac{dS}{dR}. \tag{3}$$

These equations do not determine any relations existing between w on the one hand and either u or v or both of them on the other. This was to be expected from the definitions of the fundamental operations. In view of these definitions it is evident that

$$w = f(z).$$

But there is a necessary relation between u and v, and it may be developed from the first two differential equations above. We have

$$\frac{\partial S}{\partial x} = r\frac{\partial S}{\partial R}, \quad \frac{\partial S}{\partial y} = i\frac{\partial S}{\partial R}.$$

Multiplying the first by r and the second by i and remembering that $i^2 = -r$, we obtain

$$r\frac{\partial S}{\partial x} = -i\frac{\partial S}{\partial y}.$$

Now the relation between u and v is independent of w, and hence in order to find that relation we may ignore w altogether. Geometrically, it amounts to this: For the present we are concerned with the relations between the coordinates of the projections of the variable point on the plane of reference. These are entirely independent of the point's distance from the plane; that is, of w. Therefore from the last equation we may write:

$$r\frac{\partial}{\partial x}(ur+vi) = -i\frac{\partial}{\partial y}(ur+vi).$$

$$\therefore \frac{\partial u}{\partial x}r + \frac{\partial v}{\partial x}i = -\frac{\partial u}{\partial y}i + \frac{\partial v}{\partial y}e. \qquad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are necessary relations between w and v regarded as functions of x and y. By a method entirely analogous to that employed in the ordinary theory of complex variables they may be shown to be sufficient conditions. Then we have the theorem:

THEOREM I. *In order that we may have*

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

it is necessary and sufficient that the following relations exist:

$$w=f(z),\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

Adding equations (1) and (3) we have

$$\frac{dS}{dR}r+\frac{dS}{dR}s=\frac{\partial S}{\partial x}+\frac{\partial u}{\partial z}.$$

Employing equation VI of the preceding section, this becomes

$$\frac{dS}{dR}=\frac{\partial S}{\partial x}+\frac{\partial S}{\partial z}. \tag{4}$$

(It is evident that VI holds not only when m is real but also when m is a triple number.) The second member of (4) is evidently independent of the direction in which the point $R+\triangle R$ approaches R; hence

THEOREM II. *The value of the ratio* $\frac{dS}{dR}$ *is independent of the direction in space in which* $R+\triangle R$ *approaches* R.

When in

$$ur+vi+ws=S=f(R)=f(xr+yi+zs)$$

the form of f is given, it clearly suffices theoretically to determine u, v, w. We shall show conversely that if any one of the three quantities u, v, w is given, this alone is sufficient to determine f and the other two of the functions u, v, w.

First, suppose w is given. It must be a function of z alone such that $w=f(z)$, and hence the form of f is determined. It follows as a consequence that u and v have perfectly definite forms.

On the other hand we have only to notice that the relation expressed between u and v in theorem I is exactly that which in the theory of complex variables suffices to determine one of them when the other is given. With both of them determinate, it is evident that the form of f is fixed and therefore w is perfectly definite. Hence,

THEOREM III. *If any one of the functions u, v, w (as u) is given in terms of x, y, z, or any of them, this alone suffices to determine the other two functions (as v and w) and the form of f.*

DUALITY IN THE FORMULAS OF SPHERICAL TRIGONOMETRY.

By W. A. GRANVILLE, Yale University.

It was Moebius* who first called attention to the fact that if we use the supplements of the angles of a spherical triangle instead of the angles themselves, the formulas of Spherical Trigonometry arrange themselves in pairs, either one of a pair being the dual of the other. While this very important notion is now generally employed in advanced treatises dealing with Geometry on the Sphere, I have been unable to discover any writer who has carried out this fruitful idea of duality to its logical conclusion, and that is to apply it to the solution of spherical triangles in a first course in Spherical Trigonometry.

The purpose of this paper is to call the attention of teachers to the many practical advantages attending the use of this property of duality in the formulas of Spherical Trigonometry in teaching the elements of Spherical Trigonometry.

Moebius proved his results for any spherical triangle. For the sake of brevity, however, I shall follow the ordinary practice in a first course and consider such spherical triangles only whose parts are less than 180°.

Given any relation involving one or more of the sides a, b, c, and the angles A, B, C, of any spherical triangle. Now the polar triangle (whose sides are denoted by a', b', c', and angles by A', B', C') is also a spherical triangle and the given relation must hold true for it also; that is, the given relation applies to the polar triangle if accents are placed on the letters representing the sides and angles. Thus, the First Law of Cosines, as usually given, is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{1}$$

or, for the polar triangle,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \sin A'. \tag{2}$$

But $a' = 180° - A$, $b' = 180° - B$, $c' = 180° - C$, $A' = 180° - a$.

Substituting these in (2), we get

$$\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a),$$

or (the Second Law of Cosines),

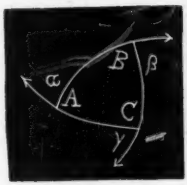
*Moebius: *Ueber eine neue Behandlungsweise der analytischen Sphaerik*, 1846. *Entwicklung der Grundformeln der Trigonometrie in grosstmoeglicher Algemeinheit*, 1860. See *Gesammelte Werke II.*

(3) $$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

which involves the sides and angles of the original triangle. Hence,

THEOREM. *In any relation between the parts of a spherical triangle each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.*

Let the supplements of the angles of the triangle be denoted by α, β, γ; *i. e.* α, β, γ are the exterior angles of the triangle. Then



$$\alpha = 180° - A,\ \beta = 180° - B,\ \gamma = 180° - C,$$
$$\text{or, } A = 180° - \alpha,\ B = 180° - \beta,\ C = 180° - \gamma.$$

When we apply the above theorem to a relation involving the sides and the supplements of the angles of a triangle, we, in fact,

replace a by α $(=180° - A)$,
replace b by β $(=180° - B)$,
replace c by γ $(=180° - C)$,
replace α $(=180° - A)$ by $180° - (180° - a) = a$,
replace β $(=180° - B)$ by $180° - (180° - b) = b$,
replace γ $(=180° - C)$ by $180° - (180° - c) = c$,

or, what amounts to the same thing, *we interchange the Greek and Roman letters.* For instance, substitute

$$A = 180° - \alpha,\ B = 180° - \beta,\ C = 180° - \gamma \text{ in (1).}$$

This gives us the First Law of Cosines in the new form

(4) $$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha.$$

Similarly, we may get

(5) $$\cos b = \cos c \cos a - \sin c \sin a \cos \beta,$$
(6) $$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$$

If we now make the substitutions indicated above in (4), (5), (6); that is, interchange the Greek and Roman letters, we get the following new form of the Second Law of Cosines:

(7) $$\cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a,$$
(8) $$\cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b,$$
(9) $$\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c;$$

that is, we have derived three new relations between the sides and the sup-

plements of the angles of the triangle. Summing up the results of our discussion, we may then state the following

PRINCIPLE OF DUALITY ON THE SPHERE. *If the sides of a spherical triangle be denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters α, β, γ, then from any given formula involving any of these six parts, we may write down a dual formula by simply interchanging the corresponding Greek and Roman letters.*

For the sake of comparison the formulas that are commonly used in the solution of oblique spherical triangles are given in the first column below, while the corresponding new forms are written in the second column.

First Law of Cosines.

$\cos a = \cos b \cos c + \sin b \sin c \cos A$	$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha$
$\cos b = \cos c \cos a + \sin c \sin a \cos B$	$\cos b = \cos c \cos a - \sin c \sin a \cos \beta$
$\cos c = \cos a \cos b + \sin a \sin b \cos C$	$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma$

Second Law of Cosines.

$\cos A = -\cos B \cos C + \sin B \sin C \cos a$	$\cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a$
$\cos B = -\cos C \cos A + \sin C \sin A \cos b$	$\cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b$
$\cos C = -\cos A \cos B + \sin A \sin B \cos c$	$\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c$

The formulas for the functions of half the supplements of the angles of a spherical triangle in terms of its sides may be derived in the same manner as the formulas for the functions of half the angles of a triangle in terms of the sides are usually derived. This gives

$\sin \frac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$	$\sin \frac{1}{2} \alpha = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$
$\cos \frac{1}{2} A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$	$\cos \frac{1}{2} \alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$
$\tan \frac{1}{2} A = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$	$\tan \frac{1}{2} \alpha = \sqrt{\frac{\sin s \sin(s-c)}{\sin(s-b)\sin(s-c)}}$

By permuting the letters in cyclical order, we get the remaining two sets of formulas of six each.

The usual method is now to derive the formulas for the functions of the half sides in terms of the angles (shown below in the first column) in like manner. By the use of the Principle of Duality, however, we get the corresponding formulas at once (as shown in the second column) by simply interchanging the Greek and Roman letters in the second column above.

$$\sin\tfrac{1}{2}a=\sqrt{\frac{-\cos A\,\cos(S-A)}{\cos B\,\cos C}} \qquad \sin\tfrac{1}{2}a=\sqrt{\frac{\sin\sigma\,\sin(\sigma-\alpha)}{\sin\beta\,\sin\gamma}}$$

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\,\sin C}} \qquad \cos\tfrac{1}{2}a=\sqrt{\frac{\sin(\sigma-\beta)\,\sin(\sigma-\gamma)}{\sin\beta\,\sin\gamma}}$$

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\,\cos(S-A)}{\cos(S-B)\cos(S-C)}} \qquad \tan\tfrac{1}{2}a=\sqrt{\frac{\sin\sigma\,\sin(\sigma-\alpha)}{\sin(\sigma-\beta)\,\sin(\sigma-\gamma)}}$$

where $s=\frac{1}{2}(a+b+c)$, $S=\frac{1}{2}(A+B+C)$, $\sigma=\frac{1}{2}(\alpha+\beta+\gamma)$.

Instead of the above two sets of formulas it is sometimes more convenient to use the following two sets when solving triangles.

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}} \qquad \tan\tfrac{1}{2}d=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan\tfrac{1}{2}A=\frac{\tan r}{\sin(s-a)} \qquad \tan\tfrac{1}{2}\alpha=\frac{\sin(s-a)}{\tan\tfrac{1}{2}d}$$

etc., where d is the diameter of the inscribed circle, and

$$\tan R=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$$

$$\tan\tfrac{1}{2}a=\tan R\,\cos(S-A)$$

$$\tan\tfrac{1}{2}\delta=\sqrt{\frac{\sin(\sigma-\alpha)\,\sin(\sigma-\beta)\,\sin(\sigma-\gamma)}{\sin\sigma}}$$

$$\tan\tfrac{1}{2}\alpha=\frac{\sin(\sigma-\alpha)}{\tan\tfrac{1}{2}\delta}$$

etc., where δ is the supplement of the diameter of the circumscribed circle.

Napier's Analogies in the old and new forms are shown below.

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad \tan\tfrac{1}{2}(\alpha+\beta)=-\frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C \qquad \tan\tfrac{1}{2}(\alpha-\beta)=-\frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\tfrac{1}{2}(A-B)}{\cos\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad \tan\tfrac{1}{2}(a+b)=-\frac{\cos\tfrac{1}{2}(\alpha-\beta)}{\cos\tfrac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\tfrac{1}{2}(A-B)}{\sin\tfrac{1}{2}(A+B)}\tan\tfrac{1}{2}c \qquad \tan\tfrac{1}{2}(a-b)=-\frac{\sin\tfrac{1}{2}(\alpha-\beta)}{\sin\tfrac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

In the case of the Law of Sines the usual form is,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

If we use the supplements of the angles this becomes

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}.$$

If we apply the Principle of Duality to the last form there results

$$\frac{\sin \alpha}{\sin a}=\frac{\sin \beta}{\sin b}=\frac{\sin \gamma}{\sin c};$$

hence, the *Law of Sines goes over into itself.*

In what follows we have in the first column the usual ten formulas used in the solution of right triangles, in terms of α, β, γ. When we apply the Principle of Duality to these we get the formulas in the second column, and these turn out to be formulas for the solution of quadrantal triangles.

Right Triangle.	*Quadrantal Triangle*
(1) $\cos c=\cos a\cos b$	$\cos\gamma=\cos\alpha\cos\beta$
(2) $\sin a=\sin c\sin\alpha$	$\sin\alpha=\sin\gamma\sin a$
(3) $\sin b=\sin c\sin\beta$	$\sin\beta=\sin\gamma\sin b$
(4) $\cos\alpha=-\cos a\sin\beta$	$\cos a=-\cos\alpha\sin b$
(5) $\cos\beta=-\cos b\sin\alpha$	$\cos b=-\cos\beta\sin a$
(6) $\cos\alpha=-\tan b\cot c$	$\cos a=-\tan\beta\cot\gamma$
(7) $\cos\beta=-\tan a\cot c$	$\cos b=-\tan\alpha\cot\gamma$
(8) $\sin b=-\tan a\cot\alpha$	$\sin\beta=-\tan\alpha\cot a$
(9) $\sin a=-\tan b\cot\beta$	$\sin\alpha=-\tan\beta\cot b$
(10) $\cos c=\cot\alpha\cot\beta$	$\cos\gamma=\cot a\cot b$

Von Staudt called the expression

$$D=\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c-2\cos a\cos b\cos c}$$

the *sine of the spherical triangle,* and

$$\Delta=\sqrt{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma}$$

the *sine of the polar triangle,* because of the relation

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma} = \frac{D}{\Delta}.$$

Other authors have called D the *first staudtian* and Δ the *second staudtian.* We see that either one goes over into the other by the application of the Principle of Duality. This is also shown by the relations

$$D = \sin b \sin c \sin \alpha = \sin c \sin a \sin \beta = \sin a \sin b \sin \gamma,$$
$$\Delta = \sin \alpha \sin \beta \sin a = \sin \beta \sin \gamma \sin b = \sin \gamma \sin \alpha \sin c.$$

Let V denote the volume of the tetrahedron whose vertices are at A, B, C, the vertices of the spherical triangle and O, the center of the sphere. Then it may be shown that

$$6V = r^3 D,$$

where r is the radius of the sphere. If we apply the Principle of Duality to this expression we get

$$6V' = r^3 \Delta,$$

where V' is the volume of the tetrahedron corresponding to the polar triangle A', B', C'.

From a theoretical standpoint the great advantages derived from using the Principle of Duality instead of the usual methods are so apparent that there is scarcely any room for argument. Nearly one-half of the work usually required in deriving the standard formulas is done away with and the resulting formulas are more easily memorized. But, the question now naturally arises, are these new formulas well adapted to numerical calculations? By actual experience in solving a large number of spherical triangles, I have found that there is little or no difference in the amount of labor involved. In the case of given angles we must of course first find their supplements before applying the new formulas, and in the case of required angles we must take the supplements of the angles found from the tables. On the other hand I have found that on account of the duality of the new formulas the detail of the work is simplified and there is less liability of making mistakes. Below is found a solution of a spherical triangle, the new formulas being used.

Example. Given $A = 70°$, $B = 131° 10'$, $C = 94° 50'$; find a, b, c.

$\alpha = 180° - A = 110°$
$\beta = 180° - B = 48° 50'$
$\gamma = 180° - C = 85° 10'$
$2\sigma = 244°$
$\sigma = 122°$
$\sigma - \alpha = 12°$
$\sigma - \beta = 73° 10'$
$\sigma - \gamma = 36° 50'$

To find $\log \tan \frac{1}{2}\delta$
$\log \sin(\sigma - \alpha) = 9.3179$
$\log \sin(\sigma - \beta) = 9.9810$
$\log \sin(\sigma - \gamma) = 9.7778$
colog $\sin \sigma = 0.0716$
$2|9.1483$
$\log \tan \frac{1}{2}\delta = 9:5742$

$\log \sin(\sigma - \alpha) = 9.3179$
$\log \tan \frac{1}{2}\delta = 9.5742$
$\log \tan \frac{1}{2}a = 9.7437$
$\frac{1}{2}a = 39°$
$a = 58°$

$\log \sin(\sigma - \beta) = 9.9810$
$\log \tan \frac{1}{2}\delta = 9.5742$
$\log \tan \frac{1}{2}b = 0.4068$
$\frac{1}{2}b = 68° 36'$
$b = 137° 12'$

$\log \sin(\sigma - \gamma) = 9.7778$
$\log \tan \frac{1}{2}\delta = 9.5742$
$\log \tan \frac{1}{2}c = 0.2036$
$\frac{1}{2}c = 57° 58'$
$c = 115° 56'$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

308. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the roots of $x^2 + px + q = 0$ may not lie between -1 and $+1$.

I. Solution by J. A. CAPARO, University of Notre Dame, Notre Dame, Ind.

In the most general case let one of the roots be $+a$ and the other $-b$, then:

$$(x-a)(x+b) = 0 \text{ or } x^2 + x(b-a) - ab = 0.$$

Comparing with $x^2 + px + q = 0$, $p = b - a$, $q = -ab$.

The conditions that the roots shall not lie between $+1$ and -1 are:

$$+a > 1 \ldots (1); \quad +b > 1 \ldots (2).$$

Multiplying, $ab > 1$, but $ab = -q$, therefore $-q > 1$ or $q < -1$. Also from (1), (2), $a - 1 > 0$, $b + 1 > 2$.

Multiplying, $ab - b + a - 1 > 0$, or $-ab + (b-a) + 1 < 0$, or $+q + p < -1$. The required conditions then are: $q < -1$ and $q + p < -1$.

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Assuming that the equation has real roots, *i. e.*, that $p^2>4q$, find Sturm's functions. They are,

$$\left.\begin{array}{l} f(x)=x^2+px+q \\ f'(x)=2x+p \\ f_2(x)=p^2-4q \end{array}\right\}$$

For $+1$, we have, $f(x)=1+p+q$, $f_1(x)=2+p$, $f_2(x)=p^2-4q$.

For -1, we have, $f(x)=1-p+q$, $f_1(x)=-2+p$, $f_2(x)=p^2-4q$.

If there is no root between $+1$ and -1, there will be the same number of variations of signs in each of these. p^2-4q is always positive. We see that the number of variations will be the same if $f(1)$ and $f(-1)$, $f'(1)$ and $f'(-1)$ have the same sign, *i. e.*, $(1+q)^2-p^2>0$ and $p^2-4>0$, *i. e.*,

$$(1+q)^2>p^2>4.$$

They will also have the same number of variations if $f'(1)$ and $f'(-1)$ have opposite signs and $f(1)$ and $f(-1)$ are both negative, *i. e.*, $p^2<4$, $1+q\pm p<0$, or $p\pm q>1$.

Also solved by G. B. M. Zerr, V. M. Spunar, and G. W. Hartwell.

309. Proposed by PROFESSOR E. B. ESCOTT, Ann Arbor, Mich.

Solve, $bx^2+cy^2+az^2=ba^2+cb^2+ac^2$,
$cx^2+ay^2+bz^2=ab^2+bc^2+ca^2$,
$xyz=abc$.

Solution by the PROPOSER.

We see by inspection that $x=a$, $y=b$, $z=c$ is one set of solutions.

Then, put $x^2=a^2+u$, $y^2=b^2+v$, $z^2=c^2+w$. The first two equations become $bu+cv+aw=0$, $cu+av+bw=0$. Solving these, we have

$$\frac{u}{a^2-bc}=\frac{v}{b^2-ca}=\frac{w}{c^2-ab}.$$

Put these equal to s, and substitute

$$\begin{array}{l} u=(a^2-bc)s, \\ v=(b^2-ca)s, \\ w=(c^2-ab)s, \end{array}$$

in values of x^2, y^2, and z^2:

$$\begin{array}{l} x^2=a^2+(a^2-bc)s, \\ y^2=b^2+(b^2-ca)s, \\ z^2=c^2+(c^2-ab)s. \end{array}$$

Substituting these values in the third of the given equations,

$$[s(a^2-bc)+a^2][s(b^2-ca)+b^2][s(c^2-ab)+c^2]=a^2b^2c^2.$$

Two of the roots of this equation are evidently $s_1=0$, $s_2=-1$. The third root can be found by subtracting the sum of these two from the sum of the roots of the equation, which is

$$\frac{a^2}{a^2-bc}+\frac{b^2}{b^2-ca}+\frac{c^2}{c^2-ab}.$$

This gives $s_3=\dfrac{a^3b^3+b^3c^3+c^3a^3-3a^2b^2c^2}{(a^2-bc)(b^2-ca)(c^2-ab)}$.

This gives the following solutions:

$x=\pm a$	$\pm\sqrt{bc}$	$\pm\sqrt{[a^2+(a^2-bc)s_3]}$
$y=\pm b$	$\pm\sqrt{ca}$	$\pm\sqrt{[b^2+(b^2-ca)s_3]}$
$z=\pm c$	$\pm\sqrt{ab}$	$\pm\sqrt{[c^2+(c^2-ab)s_3]}$

where s_3 has the value fust found.

The signs are to be taken so that xyz is positive.

Also solved by G. W. Hartwell, G. B. M. Zerr, V. M. Spunar, J. Scheffer, J. W. Clawson, and A. H. Holmes.

PROBLEMS FOR SOLUTION.

ALGEBRA.

313. Proposed by W. J. GREENTTREET, M. A., Stroud, England.

Find the conditions that the equations $px^2+qx+r=0$ and $\alpha x^2+\beta x+\gamma+y(ax^2+bx+c)=0$ may give equal values for y.

314. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Sum to infinity the series $\dfrac{1}{2.3.3.4}+\dfrac{1}{4.5.5.6}+\dfrac{1}{6.7.7.8}+\dfrac{1}{8.9.9.10}+\ldots$

315. Proposed by PROFESSOR B. F. YANNEY, Mount Union College, Alliance, Ohio.

Simplify, $1-(2-(3-\ldots-(n-1)-n)\ldots)))$.

316. Proposed by B. F. FINKEL, Ph. D.

Prove that $\sum_{r=1}^{r=n}(-1)^{n-1}\dfrac{1}{n}{}_nC_r=\dfrac{1}{1}+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}$ where ${}_nC_r=\dfrac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}$. Dickson's *College Algebra*, ex. 13, p. 92.

GEOMETRY.

343. Proposed by O. J. BROWN, Fairhope, Ala.

From any external point of a triangle, to draw a line so as to divide the triangle into two equal parts.

344. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

A tinsmith has a sheet of copper in the form of a rectangle, sides a and b. He desires to cut this into two pieces which will form a square when placed together. How can he do this?

CALCULUS.

171. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-5\left(\frac{d^2y}{dx^2}\right)\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\tfrac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

show that there is an integrating factor of the form $\left(\frac{d^2y}{dx^2}\right)^n$, and integrate the equation.

172. Proposed by CLARENCE OHLENDORF, Chicago, Ill.

Find $\int \log_e \tan^{-1} x\,dx$.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

160. Proposed by H. S. VANDIVER, Bala, Pa.

Prove that the integer next above $(\sqrt{3}+1)^{2n}$ is divisible by 2^{n+1}.

161. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find a solution of $x^4-5x^2+4\equiv 0 \pmod{p.\overline{2p+1}}$, where both p and $2p+1$ are odd primes.

162. Proposed by L. E. DICKSON, Ph. D., Associate Professor of Mathematics, The University of Chicago.

If p is an odd prime, find the number of incongruent integers x for which x^4+2ex^2+f is a quadratic residue of p.

MISCELLANEOUS.

181. Proposed by A. H. HOLMES, Brunswick, Me.

In latitude 43° 45′ N., the sun's declination being 16° 30′ N., at what time in the forenoon will the angle on the horizon between the east point and the foot of the meridian passing through the sun's position be equal to its altitude?

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FAMILIES OF CENTRAL ORBITS RELATED TO CIRCULAR TRAJECTORIES.*

By FRANK LOXLEY GRIFFIN, Williams College.

§ 1. Circles as Central Orbits.

1. The Problem. Given any central force which is everywhere finite and real, and which is a function of the distance alone, $f=f(r)$, it is known that all circles with centers at the center of force are possible orbits. For if a particle were to move in such a circle, of radius a, the centrifugal force v^2/a would have to balance exactly the attraction $f(a)$; and, conversely, if the particle were started at a distance a from the center of force, in a direction perpendicular to the radius vector, and with the velocity defined by $v^2=af(a)$, the circular orbit would result.

But in order that a circle whose center is elsewhere than at the center of force, be described as a central orbit, the force must vary according to a *special* law, which is obtainable when the polar equation of the given orbit has been written. The object of this paper is (1) to derive such a law of force, (2) to ascertain what other circular orbits are admitted by it and what properties have they in common, (3) to learn what other laws admit families of orbits having some of those properties, and (4) to point out the families in question in the case of two simple laws.

2. Laws Admitting Circular Orbits. Let the center of force be selected as the pole, and let the polar angle θ be measured from the line through the pole and the center of the circle; then the equation of the orbit is

$$2d\cos\theta=1/u-(a^2-d^2)u, \tag{1}$$

where u is the reciprocal of the radius vector, a is the radius of the circle, and d is the distance between the pole and center. [It is necessary to take

*Read before the American Mathematical Society, September 10, 1908.